

Fractal Diffraction Grating

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Abstract

We consider an optical diffraction grating in which the spatial distribution of open slits forms a fractal set. The Fraunhofer diffraction patterns through the fractal grating are obtained analytically for the simplest triad Cantor type and its generalized version. The resulting interference patterns exhibit characteristics of the original fractals and their scaling properties.

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I. INTRODUCTION

A widespread interest in fractal geometry has been generated by Mandelbrot with his monumental work [1] that deals with the geometry of phenomena observed in many fields of science. The fractal geometry is a fast growing subject and a cross-disciplinary field ranging from social science to biological science and physics [2–4].

In this article we consider another aspect of fractal geometry, namely, the Fraunhofer diffraction of an optical grating whose slit distribution is fractal. In the ordinary optical gratings the slits are evenly distributed forming a one-dimensional lattice [5]. We take, instead, a fractal grating and study the optical interference pattern by the light wave coming from the grating. As a simple example a Cantor set type grating is considered, and analytic results are obtained. The resulting interference patterns show self-similar structure and the same scaling property as those in the fractal grating itself. In other words the Fraunhofer diffraction transforms the fractal geometry of the optical grating to the fractal interference pattern.

In Sec. II the simplest triad Cantor set type grating is introduced, and the interference amplitude is explicitly evaluated. The amplitude turns out to be a repeated product of a function of a rescaled argument by 3^m for each integer m , which respects the scaling property of the diffraction grating. In Sec. III we generalize the triad Cantor set to a $(2H + 1)$ -piece Cantor set, and similar results are obtained but with a different scale factor $2H + 1$. In Sec. IV we discuss further generalizations of the model. The ordinary diffraction grating has been

an important tool in optical spectroscopy. Although we have not yet proposed any practical application, this kind of fractal diffraction grating may be useful for some purposes.

II. CANTOR DIFFRACTION GRATING

One of the simplest example of fractal sets is the Cantor set defined on a closed interval $[0, 1]$. The first stage of construction consists of dividing the interval $[0, 1]$ into three equal pieces, then removing the middle open interval, designated $(\frac{1}{3}, \frac{2}{3})$. At the second stage one divides each remaining pieces into three and again remove each middle piece. Repeating this process infinitely many times one obtains a fractal set whose fractal dimension is $D = \frac{\ln 2}{\ln 3}$ [1,2].

Let us consider a diffraction grating whose slits are distributed as the Cantor set described above. For convenience, we take the starting total transparent region to be $[-a, a]$. At the first stage we make the middle third $(-\frac{a}{3}, \frac{a}{3})$ opaque, in the second stage we again make the middle pieces of each of the two remaining thirds opaque, and so on to N times. The resulting grating consists of fractally distributed 2^N -slits of width $\frac{2a}{3^N}$. Let this be called the Cantor diffraction grating of N -th generation. Figure 1 illustrates such a construction for the case of $N = 3$. The open slits are of equal widths but unevenly distributed. The coordinates of the middle, beginning, and end points of the j -th slit are denoted by x_j, x_j^-, x_j^+ , respectively.

The Fraunhofer interference pattern of light through a Cantor diffraction grating of N -th generation can be computed as

$$\begin{aligned} A_N &= \int_{-\infty}^{\infty} dx e^{ikx \sin \theta} G(x) \\ &= \sum_{j=1}^{2^N} \frac{e^{ikx \sin \theta}}{ik \sin \theta} \Big|_{x_j^-}^{x_j^+}, \end{aligned} \quad (1)$$

where θ is the angle of light propagation with respect to the normal vector of the grating plane, and k is the wave vector. The grating function $G(x)$ is defined as

$$G(x) = \begin{cases} 0, & x \in \text{opaque region}, \\ 1, & x \in \text{open region}. \end{cases} \quad (2)$$

Using the fact that the slit width $(x_j^+ - x_j^-)$ is $\frac{2a}{3^N}$, we can write A_N as

$$A_N = \frac{2 \sin\left(\frac{ka \sin \theta}{3^N}\right)}{k \sin \theta} \sum_{j=1}^{2^N} e^{ikx_j \sin \theta}. \quad (3)$$

The amplitude A_N is factorized into two factors. The first factor, $2 \frac{\sin\left(\frac{ka \sin \theta}{3^N}\right)}{k \sin \theta}$, is just the Fraunhofer diffraction amplitude of the single slit of width $\frac{2a}{3^N}$. The second factor represents the interference of the 2^N -Cantor slits, which are expected to show the fractal nature of the slit distribution.

In order to evaluate the summation in the second factor we first notice that the slit-coordinates x_j (the midpoint of the j -th open interval) can be represented by tertiary decimal numbers where only ± 1 among $(-1, 0, +1)$ appears as numerator. Explicitly, they are

$$\begin{aligned}
x_1 &= 2a\left(\frac{-1}{3} + \frac{-1}{3^2} + \cdots + \frac{-1}{3^{N-1}} + \frac{-1}{3^N}\right), \\
x_2 &= 2a\left(\frac{-1}{3} + \frac{-1}{3^2} + \cdots + \frac{-1}{3^{N-1}} + \frac{+1}{3^N}\right), \\
x_3 &= 2a\left(\frac{-1}{3} + \frac{-1}{3^2} + \cdots + \frac{+1}{3^{N-1}} + \frac{-1}{3^N}\right), \\
&\vdots \\
x_{2^N} &= 2a\left(\frac{+1}{3} + \frac{+1}{3^2} + \cdots + \frac{+1}{3^{N-1}} + \frac{+1}{3^N}\right).
\end{aligned} \tag{4}$$

Using these coordinates we can evaluate the summation as

$$\sum_{j=1}^{2^N} e^{ikx_j \sin \theta} = \prod_{m=1}^N 2 \cos\left(\frac{2ka \sin \theta}{3^m}\right). \tag{5}$$

Substituting (5) to (3) we obtain the amplitude of the Cantor diffraction grating as

$$\begin{aligned}
A_N &= \left(\frac{2}{3}\right)^N 2a \frac{\sin\left(\frac{ka \sin \theta}{3^N}\right)}{\frac{ka \sin \theta}{3^N}} \prod_{m=1}^N \cos\left(\frac{2ka \sin \theta}{3^m}\right) \\
&= \left(\frac{2}{3}\right)^N 2a S_N(\beta) F_N(\beta),
\end{aligned} \tag{6}$$

where $\beta \equiv 2ka \sin \theta$. Here $\left(\frac{2}{3}\right)^N 2a$ signifies that the peak amplitude diminishes by the factor $\left(\frac{2}{3}\right)^N$ as the open length of the grating decreases by the same factor. The $S_N(\beta)$ is the Fraunhofer diffraction amplitude of the single slit of the width $\frac{2a}{3^N}$, and the $F_N(\beta)$ codifies the structure of the N -th generation cantor set.

The repeated product of cosine functions in the $F_N(\theta)$ has a recursive nature as

$$F_{N+1}(\beta) = \cos\left(\frac{\beta}{3}\right) F_N\left(\frac{\beta}{3}\right), \tag{7}$$

In the limit $N \rightarrow \infty$ we have

$$F_{\infty}(\beta) = \cos\left(\frac{\beta}{3}\right) F_{\infty}\left(\frac{\beta}{3}\right), \tag{8}$$

which reveals the self-similarity with respect to the scale transformation by the factor 3. The maximum of $F(\beta)$ occurs at $\beta = 0$, $F_N(0) = 1$, and the function oscillates very rapidly. Figure (2a) shows $F_N(\beta)$ in the case $2ka = 3^{10}$, $N = 10$ for $0 \leq \theta \leq \frac{\pi}{2}$, while Figure (2b) shows the same function only for the region $0 \leq \theta \leq 0.04$. Apparently they look similar, and this self-similarity becomes more striking as N increases. The distribution of zeroes looks random, which may imply that the distribution of the dark fringes in the interference pattern is fractal.

III. GENERALIZATION OF CANTOR DIFFRACTION GRATING

For the purpose of generalizing the Cantor grating we first consider a function representation of the Cantor set. Let $g(x)$ be a periodic function of period $2a$,

$$g(x) = \begin{cases} 1, & -a \leq x \leq -\frac{a}{3}, \\ 0, & -\frac{a}{3} < x < \frac{a}{3}, \\ -1, & \frac{a}{3} \leq x \leq a, \end{cases} \quad (9)$$

and define a grating structure function $G_N(x)$ as

$$G_N(x) \equiv g(x)g(3x) \cdots g(3^{N-1}x) = \prod_{m=0}^{N-1} g(3^m x), \quad (-a \leq x \leq a). \quad (10)$$

Using this grating structure function the interference pattern through the Cantor grating can be represented as

$$\begin{aligned} A_N(\beta) &= \int_{-a}^a dx e^{ikx \sin \theta} G_N(x) \\ &= \left(\frac{2}{3}\right)^N 2a S_N(\theta) F_N(\beta), \end{aligned} \quad (11)$$

where $\beta = 2ka \sin \theta$. In this way we construct a transformation between the grating function $G_N(x)$ and the interference pattern function $F_N(\beta)$ modulo the single-slit factor $S_N(\theta)$:

$$G_N(x) = \prod_{m=0}^{N-1} g(3^m x) \leftrightarrow F_N(\beta) = \prod_{m=1}^N \cos\left(\frac{\beta}{3^m}\right) = \prod_{m=0}^{N-1} \cos\left(\frac{3^m \beta}{3^N}\right), \quad (12)$$

which may be called a diffraction transformation between $G_N(x)$ and $F_N(\beta)$. Their similarity in the product form of periodic functions and scaling property are obvious. To obtain the interference pattern we simply replace the generating function $g(x)$ by the periodic function $\cos\left(\frac{\beta}{3^N}\right)$.

Let us now consider the possible generalization of the fractal structure. One immediate generalization is to take the generating function $g(x)$ different from the simplest Cantor case. For example, we consider a periodic function $h(x)$ of period $2a$. As shown in the Figure 3, the region $[-a, a]$ is divided into $2H + 1$ pieces of equal length with H being a positive integer. Its value is zero in the dark region and one in the open region, where the dark and open regions alternate.

$$h(x) = \begin{cases} 1, & -a + \frac{4ma}{2H+1} \leq x \leq -a + \frac{(4m+2)a}{2H+1}, \\ 0, & -a + \frac{(4m+2)a}{2H+1} < x < -a + \frac{4(m+1)a}{2H+1}, \end{cases} \quad (m = 0, \pm 1, \pm 2, \dots). \quad (13)$$

With $H = 1$, one finds that $h(x)$ reduces to $g(x)$ of the previous section. From this periodic function we construct a grating function

$$G_N(x) = \prod_{m=0}^{N-1} h((2H+1)^m x), \quad -a \leq x \leq a, \quad (14)$$

which, in the limit $N \rightarrow \infty$, becomes a fractal set with the fractal dimension,

$$D = \frac{\ln(H+1)}{\ln(2H+1)}. \quad (15)$$

The Fraunhofer diffraction amplitude through this grating is

$$A_N = \frac{2 \sin\left(\frac{ka \sin \theta}{(2H+1)^N}\right)}{k \sin \theta} \sum_{j=1}^{(H+1)^N} e^{ikx_j \sin \theta}, \quad (16)$$

where x_j are the midpoints of the open slits. Defining

$$F_N = \sum_{j=1}^{(H+1)^N} e^{ikx_j \sin \theta}, \quad (17)$$

we find the recursive relation

$$F_N(\beta) = \frac{\sin\left(\frac{(H+1)\beta}{(2H+1)^N}\right)}{\sin\left(\frac{\beta}{(2H+1)^N}\right)} F_{N-1}(\beta), \quad (18)$$

where $\beta = 2ka \sin \theta$. Proceeding as in the previous section, we finally get

$$F_N(\beta) = \prod_{m=0}^{N-1} \frac{\sin\left(\frac{(H+1)\beta}{(2H+1)^m}\right)}{\sin\left(\frac{\beta}{(2H+1)^m}\right)}. \quad (19)$$

In the simplest case $H = 1$, we confirm the previous result

$$F_N(\beta) = \prod_{m=1}^N \frac{\sin\left(\frac{2\beta}{3^m}\right)}{\sin\left(\frac{\beta}{3^m}\right)} = \prod_{m=1}^N 2 \cos\left(\frac{\beta}{3^m}\right). \quad (20)$$

The diffraction transformation then becomes

$$G_N(x) = \prod_{m=0}^{N-1} h\left((2H+1)^m x\right) \leftrightarrow F_N(\beta) = \prod_{m=1}^N \frac{\sin\left(\frac{(H+1)\beta}{(2H+1)^m}\right)}{\sin\left(\frac{\beta}{(2H+1)^m}\right)}. \quad (21)$$

We note that the factor function in F_N is the familiar Fraunhofer diffraction pattern of a regular grating with $(H+1)$ -slits whose width are $\frac{2a}{(2H+1)^m}$. This clearly exhibits the self-similarity with respect to the scale transformation by a factor $2H+1$.

IV. DISCUSSION

We have considered the Fraunhofer interference pattern obtained from diffraction gratings whose slits are fractally distributed. The simplest Cantor set type grating and its generalization were explicitly constructed, and their interference patterns were studied. The grating structure function, $G_N(x) = \prod_{m=0}^{N-1} g\left((2H+1)^m x\right)$, gives the interference pattern function, $F_N(\beta) = \prod_{m=0}^{N-1} f\left(\frac{\beta}{(2H+1)^m}\right)$, by the diffraction transformations, both of which show the repeated product structure, and have the self similarity with respect to the scaling by a factor of $2H+1$. However, the fractal dimension of the function of interference pattern $F_N(\beta)$ is not fully understood.

For general gratings with fractal distribution, analytic results may not be easily obtained. For example, if one tries the triadic Cantor generator whose segments are of unequal lengths,

the computations are no longer straightforward because the factorization is not possible. In this case numerical calculations would be interesting. If we consider a generator function $g(x)$ of general form, it is not easy to picture the product function $G_N(x)$ nor to evaluate the diffraction pattern function $F_N(\beta)$.

Another direction of generalization is to consider higher dimensional fractal diffraction gratings which we intend to study in future.

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V. FIGURE CAPTION

FIG. 1. Cantor Diffraction Grating for $N = 3$. The width of each open slit is $\frac{2a}{27}$. x_j, x_j^-, x_j^+ are the middle, beginning, and end points, respectively, of the j -th open slit.

FIG. 2a. The amplitude factor $F_N(x)$ with $x = 2ka \sin \theta$ for $N = 10$, $2ka = 3^{10}$, and $0 \leq \theta \leq \frac{\pi}{2}$.

FIG. 2b. The same amplitude factor $F_N(x)$, rescaled to the range $0 \leq \theta \leq 0.04$. The self similarity of the function would be more striking if N becomes larger.

FIG. 3. The Cantor-like generator of $(2H + 1)$ -slit. It is a periodic function of period $2a$. The region $[-a, a]$ is divided into $2H + 1$ pieces of equal length. The $h(x)$ takes values either zero or one alternatively.